Resit exam Linear Algebra II

Thursday 07/05/2026, 18:30–20:30

1 $((1+1)+3+(2+2)=9 \text{ pts})$ **subspaces**

Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ be subspaces of an $\mathbb{F}$-vector space $\mathcal{V}$.

(a) Show that $\mathcal{X} \cap \mathcal{Y}$ and $\mathcal{X} + \mathcal{Y}$ are subspaces of $\mathcal{V}$.

(b) Provide an example (with proof) showing that the equation

$$\mathcal{X} \cap (\mathcal{Y} + \mathcal{Z}) = (\mathcal{X} \cap \mathcal{Y}) + (\mathcal{X} \cap \mathcal{Z})$$

does *not* hold in general.

(c) Prove that

$$\mathcal{X} \cap (\mathcal{Y} + (\mathcal{X} \cap \mathcal{Z})) = (\mathcal{X} \cap \mathcal{Y}) + (\mathcal{X} \cap \mathcal{Z}).$$

2 $(2+2+3+2=9 \text{ pts})$ **bases, dimension theorem**

In the situation of exercise 1 we define a new $\mathbb{F}$-vector space denoted $\mathcal{X} \times \mathcal{Y}$ ("the cartesian product of $\mathcal{X}$ and $\mathcal{Y}$"), as follows:

$$\mathcal{X} \times \mathcal{Y} := \{(x, y) \ : \ x \in \mathcal{X},\, y \in \mathcal{Y}\},$$

with addition $(x,y)+(x',y') := (x+x', y+y')$ and scalar multiplication $c \cdot (x,y) := (cx, cy)$. You may assume without proof that indeed this defines a vector space over $\mathbb{F}$. Define the $\mathbb{F}$-linear map $T\colon \mathcal{X} \times \mathcal{Y} \to \mathcal{V}$ by $T((x,y)) = x - y$.

(a) Show that $\operatorname{Ker} T = \{(x,x) \ : \ x \in \mathcal{X} \cap \mathcal{Y}\}$.

(b) Show that $T(\mathcal{X} \times \mathcal{Y}) = \mathcal{X} + \mathcal{Y}$.

(c) Prove that if β is a basis for $\mathcal{X}$ and γ is a basis for $\mathcal{Y}$, then $\{(x,0) \ : \ x \in \beta\} \cup \{(0,y) \ : \ y \in \gamma\}$ is a basis for $\mathcal{X} \times \mathcal{Y}$.

(d) Now assume that $\mathcal{X}$ and $\mathcal{Y}$ have finite dimension over $\mathbb{F}$. Conclude from (a), (b), and (c) that $\dim_{\mathbb{F}} \mathcal{X} + \dim_{\mathbb{F}} \mathcal{Y} = \dim_{\mathbb{F}} (\mathcal{X} + \mathcal{Y}) + \dim_{\mathbb{F}} (\mathcal{X} \cap \mathcal{Y})$.

3 $(3+3+3=9$ pts) **inner product space, orthogonal complement**

We continue with the notations as in exercise 1 and 2. In addition we make the assumption that $\dim_{\mathbb{F}} \mathcal{V} = n < \infty$ and we let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathcal{V}$.

(a) Prove that $(\mathcal{X} + \mathcal{Y})^{\perp} = \mathcal{X}^{\perp} \cap \mathcal{Y}^{\perp}$.

(b) Prove that $\dim_{\mathbb{F}}(\mathcal{X}^{\perp} + \mathcal{Y}^{\perp}) = n - \dim_{\mathbb{F}}(\mathcal{X} \cap \mathcal{Y})$. *Hint:* use 2(d) and 3(a).

(c) Show that $\mathcal{X}^{\perp} + \mathcal{Y}^{\perp} = (\mathcal{X} \cap \mathcal{Y})^{\perp}$.

4 $(2+3+4=9$ pts) **positive semidefiniteness and singular values**

Let $A \in \mathbb{R}^{n \times n}$ be a real symmetric matrix.

(a) Provide an example (with proof) showing that the eigenvalues of A are, in general, not equal to the singular values of A.

(b) Show that the eigenvalues of A are nonnegative if A is positive semidefinite.

(c) Assume that A is positive semidefinite. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ be the eigenvalues of A and let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ be the singular values of A. Prove that $\lambda_i = \sigma_i$ for all $i = 1, 2, \ldots, n$.

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4 pts free